Finite Element Simulation of Coal-Bed Methane Reservoirs

Rachid Touzani
Clermont–Ferrand II, France

Laurent Alessio
Leap Energy, Kuala Lumpur, Malaysia

Introduction

CBM ?

Coalbed Methane (CBM) is a gas that occurs in association with coal. Methane is locked in coal by the water in cleats. Gas production is generally facilitated by *dewatering*. Gas migrates then through the coal matrix into the cleats. As more and more gas desorbs a two-phase flow regime develops.

The mathematical model

- Modelling of immiscible two-phase fluid flow in porous media (water + gas)
- Gas is recovered by desorption from the coalbed matrix
- Model for 2-D configurations
- Capillary pressure is neglected
- Numerical approximation by finite elements

Introduction

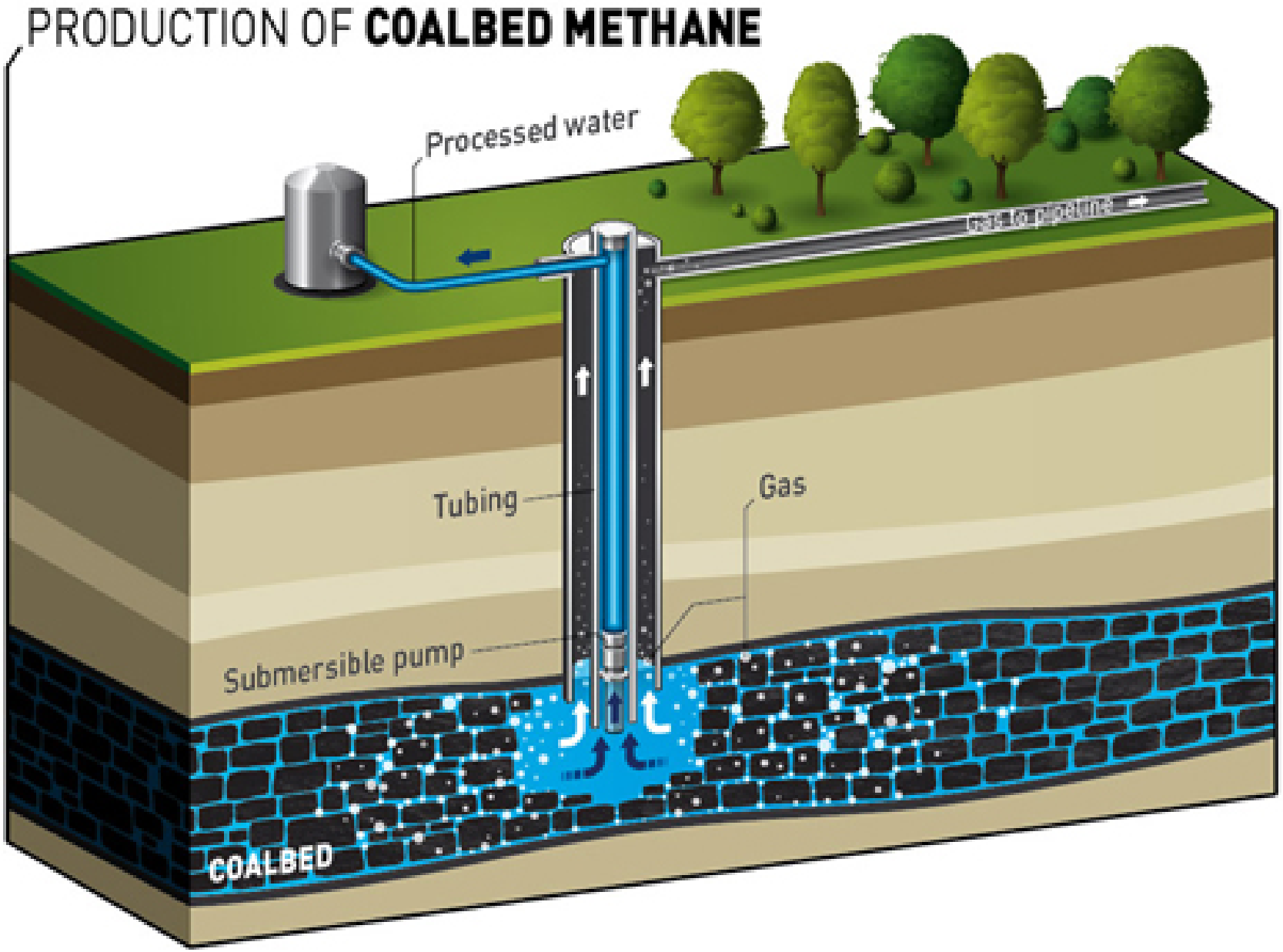
CBM ?

Coalbed Methane (CBM) is a gas that occurs in association with coal. Methane is locked in coal by the water in cleats. Gas production is generally facilitated by *dewatering*. Gas migrates then through the coal matrix into the cleats. As more and more gas desorbs a two-phase flow regime develops.

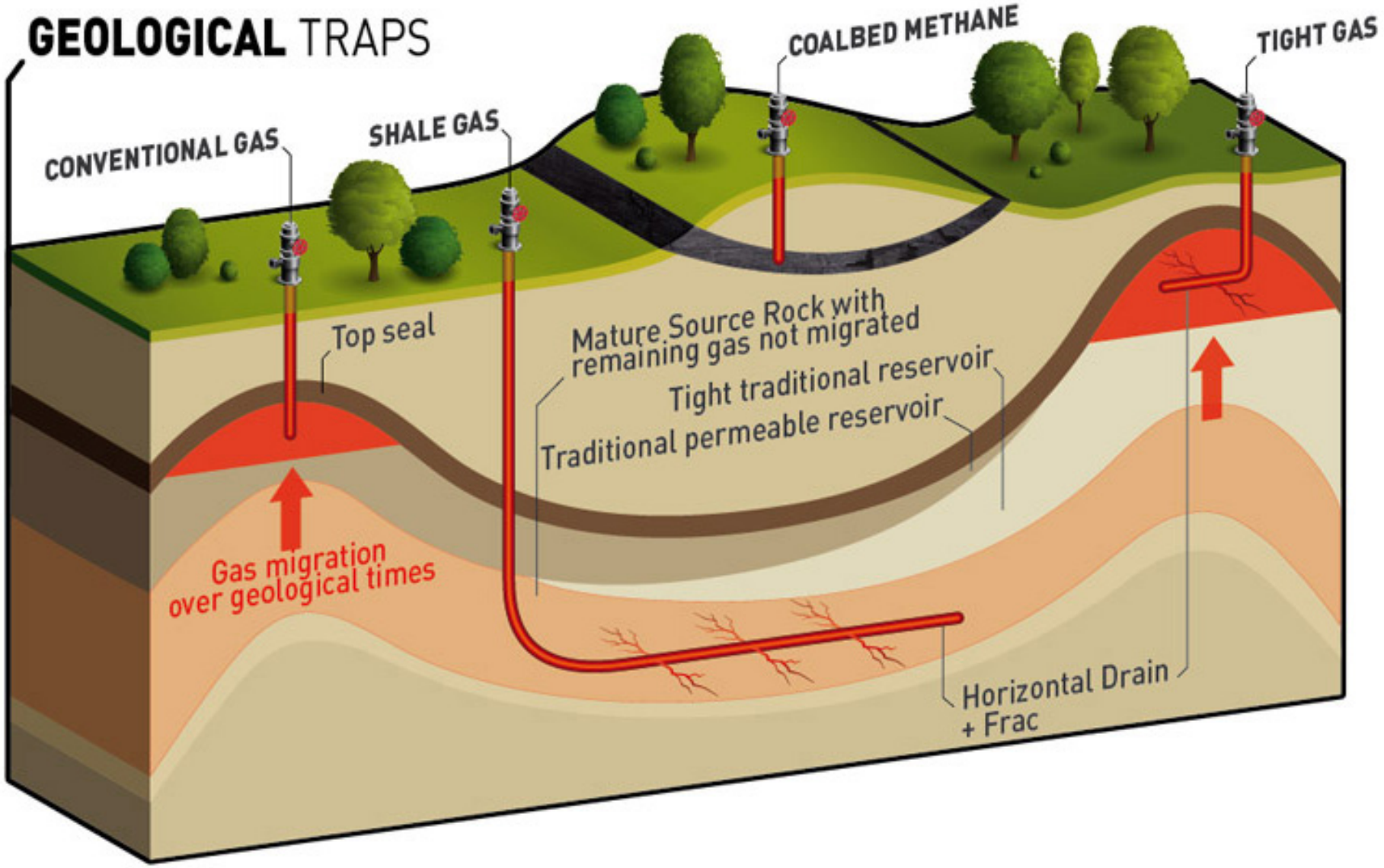
The mathematical model

- Modelling of immiscible two-phase fluid flow in porous media (water + gas)
- Gas is recovered by desorption from the coalbed matrix
- Model for 2-D configurations
- Capillary pressure is neglected
- Numerical approximation by finite elements

Principle of the CBM process



Gas recovery main procedures



The model

We consider a flow in porous medium of an immiscible mixture of water and gas.
Let S_w and S_g stand for the respective saturations of water and gas:

$$S_w + S_g = 1.$$

Mass conservation for each phase:

$$\frac{\partial}{\partial t}(\phi \varrho_w S_w) + \nabla \cdot (\varrho_w v_w) = 0$$
$$\frac{\partial}{\partial t}(\phi \varrho_g S_g) + \nabla \cdot (\varrho_g v_g) = f_D$$

where:

ϱ_w, ϱ_g	densities (water and gas)
ϕ	Porosity $(0 < \phi_0 \leq \phi(x) \leq 1)$
f_D	Rate of desorbed gas

In the following $S = S_w$ $(S_g = 1 - S)$.

The model

We consider a flow in porous medium of an immiscible mixture of water and gas.
Let S_w and S_g stand for the respective saturations of water and gas:

$$S_w + S_g = 1.$$

Mass conservation for each phase:

$$\frac{\partial}{\partial t}(\phi \varrho_w S_w) + \nabla \cdot (\varrho_w v_w) = 0$$
$$\frac{\partial}{\partial t}(\phi \varrho_g S_g) + \nabla \cdot (\varrho_g v_g) = f_D$$

where:

ϱ_w, ϱ_g	densities (water and gas)
ϕ	Porosity ($0 < \phi_0 \leq \phi(x) \leq 1$)
f_D	Rate of desorbed gas

In the following $S = S_w$ ($S_g = 1 - S$).

Darcy equation:

$$\mathbf{v}_w = -\frac{k_w}{\mu_w}\mathbf{K}\nabla p, \quad \mathbf{v}_g = -\frac{k_g}{\mu_g}\mathbf{K}\nabla p$$

where

$\mathbf{v}_w$, $\mathbf{v}_g$	Velocity of water and gas
p	Pressure (The same for both phases)
k_w, k_g	Relative permeabilities
μ_w, μ_g	Viscosities
$\mathbf{K}$	Absolute permeability tensor (assumed diagonal)

with:

$$k_w = k_w(S), \; k_g = k_g(S).$$

We define the mobilities:

$$m_w(S) = \frac{k_w(S)}{\mu_w}, \quad m_g(S,p) = \frac{k_g(S)}{\mu_g(p)},$$
$$m(S,p) = m_w(S) + m_g(S,p)$$

Darcy equation:

$$\mathbf{v}_w = -\frac{k_w}{\mu_w}\mathbf{K}\nabla p, \quad \mathbf{v}_g = -\frac{k_g}{\mu_g}\mathbf{K}\nabla p$$

where

$\mathbf{v}_w$, $\mathbf{v}_g$	Velocity of water and gas
p	Pressure (The same for both phases)
k_w, k_g	Relative permeabilities
μ_w, μ_g	Viscosities
$\mathbf{K}$	Absolute permeability tensor (assumed diagonal)

with:

$$k_w = k_w(S), \; k_g = k_g(S).$$

We define the mobilities:

$$m_w(S) = \frac{k_w(S)}{\mu_w}, \quad m_g(S,p) = \frac{k_g(S)}{\mu_g(p)},$$
$$m(S,p) = m_w(S) + m_g(S,p)$$

Darcy equation:

$$\mathbf{v}_w = -\frac{k_w}{\mu_w}\mathbf{K}\nabla p, \quad \mathbf{v}_g = -\frac{k_g}{\mu_g}\mathbf{K}\nabla p$$

where

$\mathbf{v}_w$, $\mathbf{v}_g$	Velocity of water and gas
p	Pressure (The same for both phases)
k_w, k_g	Relative permeabilities
μ_w, μ_g	Viscosities
$\mathbf{K}$	Absolute permeability tensor (assumed diagonal)

with:

$$k_w = k_w(S), \; k_g = k_g(S).$$

We define the mobilities:

$$m_w(S) = \frac{k_w(S)}{\mu_w}, \quad m_g(S,p) = \frac{k_g(S)}{\mu_g(p)},$$
$$m(S,p) = m_w(S) + m_g(S,p)$$

The total velocity is defined by

$$\mathbf{v} = \mathbf{v}_w + \mathbf{v}_g = -m(S,p)\,\mathbf{K}\nabla p.$$

We deduce the system of equations:

$$\frac{\partial}{\partial t}(\phi \varrho_w S) - \nabla \cdot \left(\varrho_w m_w \mathbf{K}\nabla p\right) = 0$$
$$\frac{\partial}{\partial t}(\phi \varrho_g (1-S)) - \nabla \cdot \left(\varrho_g m_g \mathbf{K}\nabla p\right) = f_D$$

We now assume that the water and the rock are slightly compressible, *i.e.*

$$c_w = \frac{1}{\varrho_w}\frac{d\varrho_w}{dp} = \text{Const.} > 0,$$
$$c_f = \frac{1}{\phi}\frac{d\phi}{dp} = \text{Const.} > 0,$$

In addition, we consider a real gas model:

$$\varrho_g(p) = \frac{p}{RTZ(p)}, \qquad \text{where} \quad 0 < Z(p) \leq 1.$$

The total velocity is defined by

$$\mathbf{v} = \mathbf{v}_w + \mathbf{v}_g = -m(S,p)\,\mathbf{K}\nabla p.$$

We deduce the system of equations:

$$\frac{\partial}{\partial t}(\phi \varrho_w S) - \nabla \cdot \left(\varrho_w m_w \mathbf{K}\nabla p\right) = 0$$
$$\frac{\partial}{\partial t}(\phi \varrho_g (1-S)) - \nabla \cdot \left(\varrho_g m_g \mathbf{K}\nabla p\right) = f_D$$

We now assume that the water and the rock are slightly compressible, *i.e.*

$$c_w = \frac{1}{\varrho_w}\frac{d\varrho_w}{dp} = \text{Const.} > 0,$$
$$c_f = \frac{1}{\phi}\frac{d\phi}{dp} = \text{Const.} > 0,$$

In addition, we consider a real gas model:

$$\varrho_g(p) = \frac{p}{RTZ(p)}, \qquad \text{where} \quad 0 < Z(p) \leq 1.$$

The total velocity is defined by

$$\mathbf{v} = \mathbf{v}_w + \mathbf{v}_g = -m(S,p)\,\mathbf{K}\nabla p.$$

We deduce the system of equations:

$$\frac{\partial}{\partial t}(\phi \varrho_w S) - \nabla \cdot \left(\varrho_w m_w \mathbf{K}\nabla p\right) = 0$$
$$\frac{\partial}{\partial t}(\phi \varrho_g (1-S)) - \nabla \cdot \left(\varrho_g m_g \mathbf{K}\nabla p\right) = f_D$$

We now assume that the water and the rock are slightly compressible, *i.e.*

$$c_w = \frac{1}{\varrho_w}\frac{d\varrho_w}{dp} = \text{Const.} > 0,$$
$$c_f = \frac{1}{\phi}\frac{d\phi}{dp} = \text{Const.} > 0,$$

In addition, we consider a real gas model:

$$\varrho_g(p) = \frac{p}{RTZ(p)}, \qquad \text{where} \quad 0 < Z(p) \leq 1.$$

The total velocity is defined by

$$\mathbf{v} = \mathbf{v}_w + \mathbf{v}_g = -m(S,p)\,\mathbf{K}\nabla p.$$

We deduce the system of equations:

$$\frac{\partial}{\partial t}(\phi \varrho_w S) - \nabla \cdot \left(\varrho_w m_w \mathbf{K}\nabla p\right) = 0$$
$$\frac{\partial}{\partial t}(\phi \varrho_g (1-S)) - \nabla \cdot \left(\varrho_g m_g \mathbf{K}\nabla p\right) = f_D$$

We now assume that the water and the rock are slightly compressible, *i.e.*

$$c_w = \frac{1}{\varrho_w}\frac{d\varrho_w}{dp} = \text{Const.} > 0,$$
$$c_f = \frac{1}{\phi}\frac{d\phi}{dp} = \text{Const.} > 0,$$

In addition, we consider a real gas model:

$$\varrho_g(p) = \frac{p}{RTZ(p)}, \qquad \text{where} \quad 0 < Z(p) \leq 1.$$

We thus obtain

$$\frac{\partial \varrho_w}{\partial t} = c_w \varrho_w \frac{\partial p}{\partial t}, \quad \nabla \varrho_w = c_w \varrho_w \nabla p,$$
$$\frac{\partial \phi}{\partial t} = c_f \phi \frac{\partial p}{\partial t}, \qquad \nabla \phi = c_f \phi \nabla p.$$

Neglecting nonlinear quadratic terms and dividing by ϱ_w the water conservation equation gives

$$\phi \frac{\partial S}{\partial t} + (c_w + c_f)\phi S \frac{\partial p}{\partial t} - \nabla \cdot (m_w \mathbf{K} \nabla p) = 0$$

The equation of gas becomes:

$$\frac{\partial}{\partial t}(\phi(1-S)\varrho_g) = \phi(1-S)\frac{\partial \varrho_g}{\partial t} + \phi\varrho_g(1-S)\frac{\partial \phi}{\partial t} - \phi\varrho_g\frac{\partial S}{\partial t}$$
$$= \varrho_g(1-S)(c_g + c_f)\phi\frac{\partial p}{\partial t} - \phi\varrho_g\frac{\partial S}{\partial t}$$

where

$$c_g(p) = \frac{1}{\varrho_g}\frac{d\varrho_g}{dp} = \frac{1}{RT}\frac{Z(p) - Z'(p)p}{pZ^2(p)}.$$

Neglecting nonlinear quadratic terms and dividing by ϱ_g we get:

$$-\phi\frac{\partial S}{\partial t} + (1-S)(c_g + c_f)\phi\frac{\partial p}{\partial t} - \nabla\cdot(m_g\mathbf{K}\nabla p) = \frac{f_D}{\varrho_g}$$

The equation of gas becomes:

$$\frac{\partial}{\partial t}(\phi(1-S)\varrho_g) = \phi(1-S)\frac{\partial \varrho_g}{\partial t} + \phi\varrho_g(1-S)\frac{\partial \phi}{\partial t} - \phi\varrho_g\frac{\partial S}{\partial t}$$
$$= \varrho_g(1-S)(c_g + c_f)\phi\frac{\partial p}{\partial t} - \phi\varrho_g\frac{\partial S}{\partial t}$$

where

$$c_g(p) = \frac{1}{\varrho_g}\frac{d\varrho_g}{dp} = \frac{1}{RT}\frac{Z(p) - Z'(p)p}{pZ^2(p)}.$$

Neglecting nonlinear quadratic terms and dividing by ϱ_g we get:

$$-\phi\frac{\partial S}{\partial t} + (1-S)(c_g + c_f)\phi\frac{\partial p}{\partial t} - \nabla\cdot(m_g\mathbf{K}\nabla p) = \frac{f_D}{\varrho_g}$$

Modelling of desorption

Denoting by V the adsorbed gas volume, we have the source term

$$f_D = -\varrho_m \varrho_b \frac{\partial V}{\partial t}.$$

where ϱ_m and ϱ_b are the Methane and bulk densities.

We have, at equilibrium, the Langmuir isotherm:

$$V = \frac{V_L p}{p_L + p},$$

where:

- p_L: Langmuir adsorption constant
- V_L: Available gas volume

Modelling of desorption

Denoting by V the adsorbed gas volume, we have the source term

$$f_D = -\varrho_m \varrho_b \frac{\partial V}{\partial t}.$$

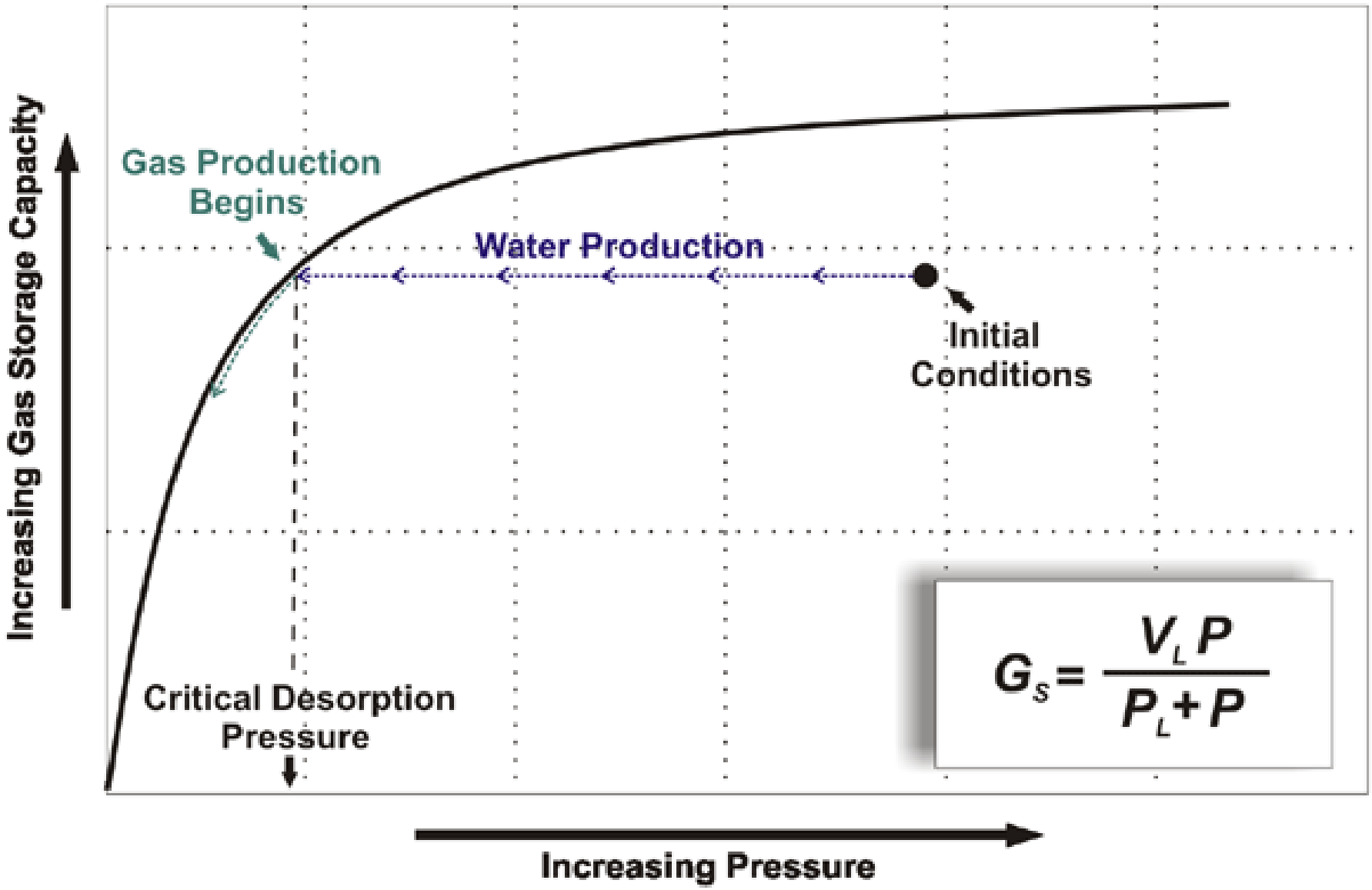
where ϱ_m and ϱ_b are the Methane and bulk densities.

We have, at equilibrium, the Langmuir isotherm:

$$V = \frac{V_L p}{p_L + p},$$

where:

- p_L: Langmuir adsorption constant
- V_L: Available gas volume


Increasing Gas Storage Capacity
Gas Production Begins
Water Production
Initial Conditions
Critical Desorption Pressure
Increasing Pressure
$G_s = \frac{V_L P}{P_L + P}$

In a thermodynamical nonequilibrium situation, we have

$$\frac{\partial V}{\partial t} = -\frac{1}{\tau}\left(V - \frac{V_L p}{p_L + p}\right)$$

where $\tau > 0$ is a characteristic diffusion time.

The complete model

We have the system:

$$\phi\frac{\partial S}{\partial t} + (c_w\phi + c_f)S\,\frac{\partial p}{\partial t} - \nabla\cdot(m_w\mathbf{K}\nabla p) = 0$$
$$-\phi\frac{\partial S}{\partial t} + (1-S)(c_g + c_f)\phi\frac{\partial p}{\partial t} - \nabla\cdot(m_g\mathbf{K}\nabla p) = \frac{\varrho_m\varrho_b}{\tau\varrho_g}\left(V - \frac{V_L p}{p_L + p}\right)$$
$$\frac{\partial V}{\partial t} + \frac{1}{\tau}\left(V - \frac{V_L p}{p_L + p}\right) = 0.$$

We can resort to the so-called *Peaceman* formulation: Adding these two equations we obtain:

$$\phi\frac{\partial S}{\partial t} + (c_w\phi + c_f)S\,\frac{\partial p}{\partial t} - \nabla\cdot(m_w\mathbf{K}\nabla p) = 0$$
$$c_t\phi\frac{\partial p}{\partial t} - \nabla\cdot(m\mathbf{K}\nabla p) = \frac{\varrho_m\varrho_b}{\tau\varrho_g}\left(V - \frac{V_L p}{p_L + p}\right)$$
$$\frac{\partial V}{\partial t} + \frac{1}{\tau}\left(V - \frac{V_L p}{p_L + p}\right) = 0$$

where $c_t = c_w S + c_g(1-S) + c_f$ is the total compressibility coefficient.

We have the system:

$$\phi\frac{\partial S}{\partial t}+(c_w\phi+c_f)S\,\frac{\partial p}{\partial t}-\nabla\cdot(m_w\mathbf{K}\nabla p)=0$$
$$-\phi\frac{\partial S}{\partial t}+(1-S)(c_g+c_f)\phi\frac{\partial p}{\partial t}-\nabla\cdot\left(m_g\mathbf{K}\nabla p\right)=\frac{\varrho_m\varrho_b}{\tau\varrho_g}\left(V-\frac{V_L p}{p_L+p}\right)$$
$$\frac{\partial V}{\partial t}+\frac{1}{\tau}\left(V-\frac{V_L p}{p_L+p}\right)=0.$$

We can resort to the so-called *Peaceman* formulation: Adding these two equations we obtain:

$$\phi\frac{\partial S}{\partial t}+(c_w\phi+c_f)S\,\frac{\partial p}{\partial t}-\nabla\cdot(m_w\mathbf{K}\nabla p)=0$$
$$c_t\phi\frac{\partial p}{\partial t}-\nabla\cdot\left(m\mathbf{K}\nabla p\right)=\frac{\varrho_m\varrho_b}{\tau\varrho_g}\left(V-\frac{V_L p}{p_L+p}\right)$$
$$\frac{\partial V}{\partial t}+\frac{1}{\tau}\left(V-\frac{V_L p}{p_L+p}\right)=0$$

where $c_t=c_wS+c_g(1-S)+c_f$ is the total compressibility coefficient.

Remarks

1. The main advantage of this formulation is that

$$m \geq m_0 > 0 \quad \text{although } m_g \geq 0, \quad m_w \geq 0.$$

   i.e. the equation does not degenerate.

2. Although the Peaceman formulation seems attractive, its performance, from numerical point of view, gives poor accuracy. We shall however keep here its presentation for its relative simplicity.

3. We have

$$\nabla \cdot (m_w \mathbf{K} \nabla p) = m_w \nabla \cdot (\mathbf{K} \nabla p) + \mathbf{K} \nabla p \cdot \nabla m_w = m_w \nabla \cdot (\mathbf{K} \nabla p) + m_w'(S) \mathbf{K} \nabla p \cdot \nabla S$$

   which is a transport problem (for given p). This implies the necessity of using an upwind scheme, if the capillary pressure is null (or small enough).

Remarks

1. The main advantage of this formulation is that

$$m \geq m_0 > 0 \quad \text{although } m_g \geq 0, \quad m_w \geq 0.$$

   i.e. the equation does not degenerate.

2. Although the Peaceman formulation seems attractive, its performance, from numerical point of view, gives poor accuracy. We shall however keep here its presentation for its relative simplicity.

3. We have

$$\nabla \cdot (m_w \mathbf{K} \nabla p) = m_w \nabla \cdot (\mathbf{K} \nabla p) + \mathbf{K} \nabla p \cdot \nabla m_w = m_w \nabla \cdot (\mathbf{K} \nabla p) + m_w'(S) \mathbf{K} \nabla p \cdot \nabla S$$

   which is a transport problem (for given p). This implies the necessity of using an upwind scheme, if the capillary pressure is null (or small enough).

Remarks

1. The main advantage of this formulation is that

$$m \geq m_0 > 0 \quad \text{although } m_g \geq 0, \quad m_w \geq 0.$$

   i.e. the equation does not degenerate.

2. Although the Peaceman formulation seems attractive, its performance, from numerical point of view, gives poor accuracy. We shall however keep here its presentation for its relative simplicity.

3. We have

$$\nabla \cdot (m_w \mathbf{K} \nabla p) = m_w \nabla \cdot (\mathbf{K} \nabla p) + \mathbf{K} \nabla p \cdot \nabla m_w = m_w \nabla \cdot (\mathbf{K} \nabla p) + m_w'(S) \mathbf{K} \nabla p \cdot \nabla S$$

   which is a transport problem (for given p). This implies the necessity of using an upwind scheme, if the capillary pressure is null (or small enough).

Well modelling

In realistic situations, the domain (reservoir) contains wells with a small diameter (with respect to the reservoir's diameter). This is generally at the origin of serious numerical difficulties.

Consider, for instance, the case of a vertical well. We assume that the flow is radial in the vicinity of the well. We also assume that the flow is incompressible in this neighborhood and has constant properties.

The flow is then modelled in this neighborhood, for the water phase by

$$-\nabla \cdot (\varrho_w m_w \mathbf{K} \nabla p) = \frac{q_w}{H} \delta$$

where δ is the Dirac distribution at the center of the well, q_w is the well's production rate for the water and H is the reservoir's height.

We obtain the analytical solution

$$p(r) = p(r_w) - \frac{q_w}{2\pi \varrho_w m_w \kappa H} \ln\left(\frac{r}{r_w}\right), \qquad r = (x_1^2 + x_2^2)^{\frac{1}{2}} - r_w$$

where r_w is the well radius and $\kappa = \sqrt{K_{11} K_{22}}$.

Well modelling

In realistic situations, the domain (reservoir) contains wells with a small diameter (with respect to the reservoir's diameter). This is generally at the origin of serious numerical difficulties.

Consider, for instance, the case of a vertical well. We assume that the flow is radial in the vicinity of the well. We also assume that the flow is incompressible in this neighborhood and has constant properties.

The flow is then modelled in this neighborhood, for the water phase by

$$-\nabla \cdot (\varrho_w m_w \mathbf{K} \nabla p) = \frac{q_w}{H} \delta$$

where δ is the Dirac distribution at the center of the well, q_w is the well's production rate for the water and H is the reservoir's height.

We obtain the analytical solution

$$p(r) = p(r_w) - \frac{q_w}{2\pi \varrho_w m_w \kappa H} \ln\left(\frac{r}{r_w}\right), \qquad r = (x_1^2 + x_2^2)^{\frac{1}{2}} - r_w$$

where r_w is the well radius and $\kappa = \sqrt{K_{11} K_{22}}$.

Well modelling

In realistic situations, the domain (reservoir) contains wells with a small diameter (with respect to the reservoir's diameter). This is generally at the origin of serious numerical difficulties.

Consider, for instance, the case of a vertical well. We assume that the flow is radial in the vicinity of the well. We also assume that the flow is incompressible in this neighborhood and has constant properties.

The flow is then modelled in this neighborhood, for the water phase by

$$-\nabla \cdot (\varrho_w m_w \mathbf{K} \nabla p) = \frac{q_w}{H} \delta$$

where δ is the Dirac distribution at the center of the well, q_w is the well's production rate for the water and H is the reservoir's height.

We obtain the analytical solution

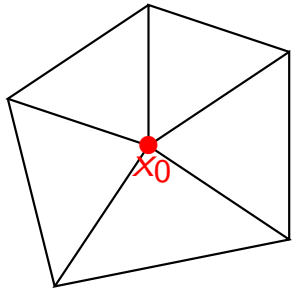
$$p(r) = p(r_w) - \frac{q_w}{2\pi \varrho_w m_w \kappa H} \ln\left(\frac{r}{r_w}\right), \qquad r = (x_1^2 + x_2^2)^{\frac{1}{2}} - r_w$$

where r_w is the well radius and $\kappa = \sqrt{K_{11} K_{22}}$.

Let φ_0 denote the $\mathbb{P}_1$–basis function at node x_0 (well node), we have

$$\varrho_w m_w H \sum_{e \subset \Omega_0} \int_e \mathbf{K} \nabla p \cdot \varphi_0 \, dx = q_w$$

where Ω_0 is the support de φ_0.



We assume that the analytical solution is a good approximation of the pressure at neighboring nodes.

Using the expansion

$$p = \sum_i p_i \varphi_i \qquad \text{in } \Omega_0$$

we get

$$\varrho_w m_w H \sum_{e \subset \Omega_0} \sum_i \Big(\int_e \mathbf{K} \varphi_i \cdot \nabla \varphi_0 \, dx \Big) p_i = q_w.$$

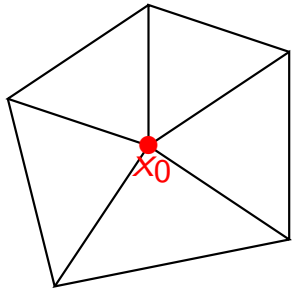
Then

$$\varrho_w m_w H \sum_{i \neq 0} T_i (p_i - p_0) = q_w \quad \text{where } T_i = \sum_{\ell=1}^{2} \int_{e_\ell} \mathbf{K} \nabla \varphi_i \cdot \varphi_0 \, dx$$

Let φ_0 denote the $\mathbb{P}_1$–basis function at node x_0 (well node), we have

$$\varrho_w m_w H \sum_{e \subset \Omega_0} \int_e \mathbf{K} \nabla p \cdot \varphi_0 \, dx = q_w$$

where Ω_0 is the support de φ_0.



We assume that the analytical solution is a good approximation of the pressure at neighboring nodes.

Using the expansion

$$p = \sum_i p_i \varphi_i \qquad \text{in } \Omega_0$$

we get

$$\varrho_w m_w H \sum_{e \subset \Omega_0} \sum_i \Big(\int_e \mathbf{K} \varphi_i \cdot \nabla \varphi_0 \, dx \Big) p_i = q_w .$$

Then

$$\varrho_w m_w H \sum_{i \neq 0} T_i (p_i - p_0) = q_w \quad \text{where } T_i = \sum_{\ell=1}^{2} \int_{e_\ell} \mathbf{K} \nabla \varphi_i \cdot \varphi_0 \, dx$$

Using the analytical solution, we obtain

Well model for the water phase

$$q_w = \frac{\sum_{i \neq 0} T_i}{1 + \frac{1}{2\pi\kappa} \sum_{i \neq 0} T_i \ln(r_i / r_w)} \varrho_w m_w H (p_b - p_0)$$

For the gas phase, the situation is more delicate: One cannot assume that ϱ_g is constant in the vicinity of a well.
We use the Kirchhoff transformation by defining

$$\tilde{p} = \int_{p_0}^{p} \varrho_g(s)\, ds$$

Then, we have

$$-m_g \kappa \Delta \tilde{p} = q_g \delta$$

the solution of which is given by

$$\tilde{p}(r) = \tilde{p}(r_w) - \frac{q_g}{2\pi m_g \kappa H} \ln\left(\frac{r}{r_w}\right)$$

Using the analytical solution, we obtain

Well model for the water phase

$$q_w = \frac{\sum_{i \neq 0} T_i}{1 + \frac{1}{2\pi\kappa} \sum_{i \neq 0} T_i \ln(r_i / r_w)} \varrho_w m_w H (p_b - p_0)$$

For the gas phase, the situation is more delicate: One cannot assume that ϱ_g is constant in the vicinity of a well.
We use the Kirchhoff transformation by defining

$$\tilde{p} = \int_{p_0}^{p} \varrho_g(s)\, ds$$

Then, we have

$$-m_g \kappa \Delta \tilde{p} = q_g \delta$$

the solution of which is given by

$$\tilde{p}(r) = \tilde{p}(r_w) - \frac{q_g}{2\pi m_g \kappa H} \ln\left(\frac{r}{r_w}\right)$$

Using the analytical solution, we obtain

Well model for the water phase

$$q_w = \frac{\sum_{i\neq 0} T_i}{1 + \frac{1}{2\pi\kappa}\sum_{i\neq 0} T_i \ln(r_i/r_w)} \varrho_w m_w H(p_b - p_0)$$

For the gas phase, the situation is more delicate: One cannot assume that ϱ_g is constant in the vicinity of a well.
We use the Kirchhoff transformation by defining

$$\tilde{p} = \int_{p_0}^{p} \varrho_g(s)\, ds$$

Then, we have

$$-m_g \kappa \Delta \tilde{p} = q_g \delta$$

the solution of which is given by

$$\tilde{p}(r) = \tilde{p}(r_w) - \frac{q_g}{2\pi m_g \kappa H} \ln\left(\frac{r}{r_w}\right)$$

We use a $\mathbb{P}_1$–finite element method with a *Streamline Upwind* stabilization term:
Let $\mathscr{T}(\Omega)$ denote a triangulation of Ω and let us define the finite dimensional space:

$$\mathcal{S} = \{\psi \in \mathcal{C}^0(\overline{\Omega});\ \psi_{|K} \in \mathbb{P}_1\ \forall\ K \in \mathscr{T}(\Omega)\},$$
$$\mathcal{P} = \{q \in \mathcal{C}^0(\overline{\Omega});\ q_{|K} \in \mathbb{P}_1\ \forall\ K \in \mathscr{T}(\Omega)\},$$
$$\mathcal{V} = \{W;\ W_{|K} = \text{Const.}\ \forall\ K \in \mathscr{T}(\Omega)\}.$$

Streamline Upwind stabilization

Consider the diffusion-convection equation

$$-\varepsilon \Delta u + \mathbf{a} \cdot \nabla u = f \qquad \text{in } \Omega$$

It is well known that if the local Péclet number

$$Pe = \frac{|\mathbf{a}| h}{2\varepsilon} > 1$$

then a standard (centered) discretization leads to a nonmonotone matrix and then to instabilities.

To remedy to this, a Petrov-Galerkin formulation has been proposed in the 80's by T.J.R. Hughes *et al.* and analyzed by C. Johnson. It consists in the following variational formulation:

$$\int_\Omega \varepsilon \nabla u_h \cdot \nabla v \, dx + \int_\Omega (\mathbf{a} \cdot \nabla u_h) v \, dx + \sum_K \frac{h_K}{2|\mathbf{a}|} \int_K (\mathbf{a} \cdot \nabla u_h)(\mathbf{a} \cdot \nabla v) \, dx = \int_\Omega f v \, dx \quad \forall \, v \in \mathcal{V}_h$$

Streamline Upwind stabilization

Consider the diffusion-convection equation

$$-\varepsilon \Delta u + \mathbf{a} \cdot \nabla u = f \qquad \text{in } \Omega$$

It is well known that if the local Péclet number

$$Pe = \frac{|\mathbf{a}| h}{2\varepsilon} > 1$$

then a standard (centered) discretization leads to a nonmonotone matrix and then to instabilities.

To remedy to this, a Petrov-Galerkin formulation has been proposed in the 80's by T.J.R. Hughes *et al.* and analyzed by C. Johnson. It consists in the following variational formulation:

$$\int_\Omega \varepsilon \nabla u_h \cdot \nabla v \, dx + \int_\Omega (\mathbf{a} \cdot \nabla u_h) v \, dx + \sum_K \frac{h_K}{2|\mathbf{a}|} \int_K (\mathbf{a} \cdot \nabla u_h)(\mathbf{a} \cdot \nabla v) \, dx = \int_\Omega f v \, dx \quad \forall \ v \in \mathscr{V}_h$$

We define the variational formulation (We keep the same notation for the unknowns and their approximations):
We seek $S(\cdot,t) \in \mathcal{S}$, $p(\cdot,t) \in \mathcal{P}$ and $V(t) \in \mathcal{V}$ such that for all $\varphi \in \mathcal{S}$ and $\psi \in \mathcal{P}$ (assuming Neumann boundary condition for pressure):

$$\int_\Omega \phi \frac{\partial S}{\partial t} \varphi \, dx + \int_\Omega (c_w \phi + c_f) S \frac{\partial p}{\partial t} \varphi \, dx + \int_\Omega m_w \mathbf{K} \nabla p \cdot \nabla \varphi \, dx$$

$$+ \sum_{K \in \mathscr{T}(\Omega)} \xi_K \int_K (\mathbf{K} \nabla p \cdot \nabla S)(\mathbf{K} \nabla p \cdot \nabla \varphi) \, dx = - \sum_{i=1}^{n_w} \frac{q_{wi}}{H} \varphi(x_{wi})$$

$$\int_\Omega c_t \phi \frac{\partial p}{\partial t} \psi \, dx + \int_\Omega m \mathbf{K} \nabla p \cdot \nabla \psi \, dx$$

$$= \frac{\varrho_m \varrho_b}{\tau} \int_\Omega \frac{1}{\varrho_g} \left(V - \frac{V_L \, p}{p_L + p} \right) \psi \, dx - \sum_{i=1}^{n_w} \frac{q_{gi}}{H} \varphi(x_{wi})$$

$$\frac{\partial V}{\partial t} + \frac{1}{\tau \varrho_g} \left(\frac{V_L \, p}{p_L + p} - V \right) = 0$$

with

$$\xi_K = \frac{h_K}{2 |\mathbf{K} \nabla p|} |m_w'(S)|$$

We choose the implicit Euler scheme:

$$\frac{1}{\delta t}\int_\Omega \phi^{n+1}(S^{n+1}-S^n)\varphi\,dx + \frac{1}{\delta t}\int_\Omega (c_w\phi^{n+1}+c_f^{n+1})(p^{n+1}-p^n)\varphi\,dx$$
$$+\int_\Omega m_w^{n+1}\mathbf{K}\nabla p^{n+1}\cdot\nabla\varphi\,dx$$
$$+\sum_{K\in\mathscr{T}(\Omega)}\xi_K^n\int_K(\mathbf{K}\nabla p^n\cdot\nabla S^{n+1})(\mathbf{K}\nabla p^n\cdot\nabla\varphi)\,dx = -\sum_{i=1}^{n_w}\frac{q_{wi}^{n+1}}{H}\varphi(x_{wi})$$

$$\frac{1}{\delta t}\int_\Omega c_t^{n+1}\phi^{n+1}(p^{n+1}-p^n)\psi\,dx + \int_\Omega m^{n+1}\mathbf{K}\nabla p^{n+1}\cdot\nabla\psi\,dx$$
$$= \frac{\varrho_m\varrho_b}{\tau+\delta t}\int_\Omega\frac{1}{\varrho_g^{n+1}}\left(V^n-\frac{V_L\,p^{n+1}}{p_L+p^{n+1}}\right)\psi\,dx - \sum_{i=1}^{n_w}\frac{q_{gi}^{n+1}}{H}\varphi(x_{wi})$$

$$V^{n+1} = \frac{1}{\tau+\delta t}\left(\tau V^n+\delta t\frac{V_L\,p^{n+1}}{p_L+p^{n+1}}\right)$$

for all $\varphi\in\mathcal{S}$ and $\psi\in\mathcal{P}_0$.

Note that the variable V is decoupled from S and p.

Adaptive time stepping

For numerical simulations, we resort two adaptive time stepping, for at least 2 reasons

- There is a singularity in initial data, due to the fact that we start with a saturation state ($S = 1$).
- Realistic simulations require sudden changes of prescribed well pressures.

Then, in order to optimize the computational time, an adaptive time stepping procedure is used. We use the following procedure:

For all n, we compute

$$\alpha^n = \frac{\delta t^n}{\varepsilon}\left(\frac{\|p^{n+1} - p^n\|}{\|p^n\|} + \frac{\|S^{n+1} - S^n\|}{\|S^n\|}\right)$$

We choose

$$\delta t^{n+1} = \begin{cases} \min\left(\theta, \dfrac{\alpha^n}{\delta t_n}\right)\delta t^n & \text{if } \alpha^n > \delta t^n \\ \dfrac{\delta t^n}{\min\left(\theta, \dfrac{\alpha^n}{\delta t^n}\right)} & \text{if } \alpha^n \leq \delta t^n \end{cases}$$

where ε is a given tolerance and θ is the maximal (given) value of $\delta t^{n+1}/\delta t^n$ or $\delta t^n/\delta t^{n+1}$.

Adaptive time stepping

For numerical simulations, we resort two adaptive time stepping, for at least 2 reasons

- There is a singularity in initial data, due to the fact that we start with a saturation state ($S = 1$).
- Realistic simulations require sudden changes of prescribed well pressures.

Then, in order to optimize the computational time, an adaptive time stepping procedure is used. We use the following procedure:

For all n, we compute

$$\alpha^n = \frac{\delta t^n}{\varepsilon}\left(\frac{\|p^{n+1} - p^n\|}{\|p^n\|} + \frac{\|S^{n+1} - S^n\|}{\|S^n\|}\right)$$

We choose

$$\delta t^{n+1} = \begin{cases} \min\left(\theta, \dfrac{\alpha^n}{\delta t_n}\right)\delta t^n & \text{if } \alpha^n > \delta t^n \\ \dfrac{\delta t^n}{\min\left(\theta, \dfrac{\alpha^n}{\delta t^n}\right)} & \text{if } \alpha^n \leq \delta t^n \end{cases}$$

where ε is a given tolerance and θ is the maximal (given) value of $\delta t^{n+1}/\delta t^n$ or $\delta t^n/\delta t^{n+1}$.

Adaptive time stepping

For numerical simulations, we resort two adaptive time stepping, for at least 2 reasons

- There is a singularity in initial data, due to the fact that we start with a saturation state ($S = 1$).
- Realistic simulations require sudden changes of prescribed well pressures.

Then, in order to optimize the computational time, an adaptive time stepping procedure is used. We use the following procedure:

For all n, we compute

$$\alpha^n = \frac{\delta t^n}{\varepsilon}\left(\frac{\|p^{n+1} - p^n\|}{\|p^n\|} + \frac{\|S^{n+1} - S^n\|}{\|S^n\|}\right)$$

We choose

$$\delta t^{n+1} = \begin{cases} \min\left(\theta, \dfrac{\alpha^n}{\delta t_n}\right)\delta t^n & \text{if } \alpha^n > \delta t^n \\ \dfrac{\delta t^n}{\min\left(\theta, \dfrac{\alpha^n}{\delta t^n}\right)} & \text{if } \alpha^n \leq \delta t^n \end{cases}$$

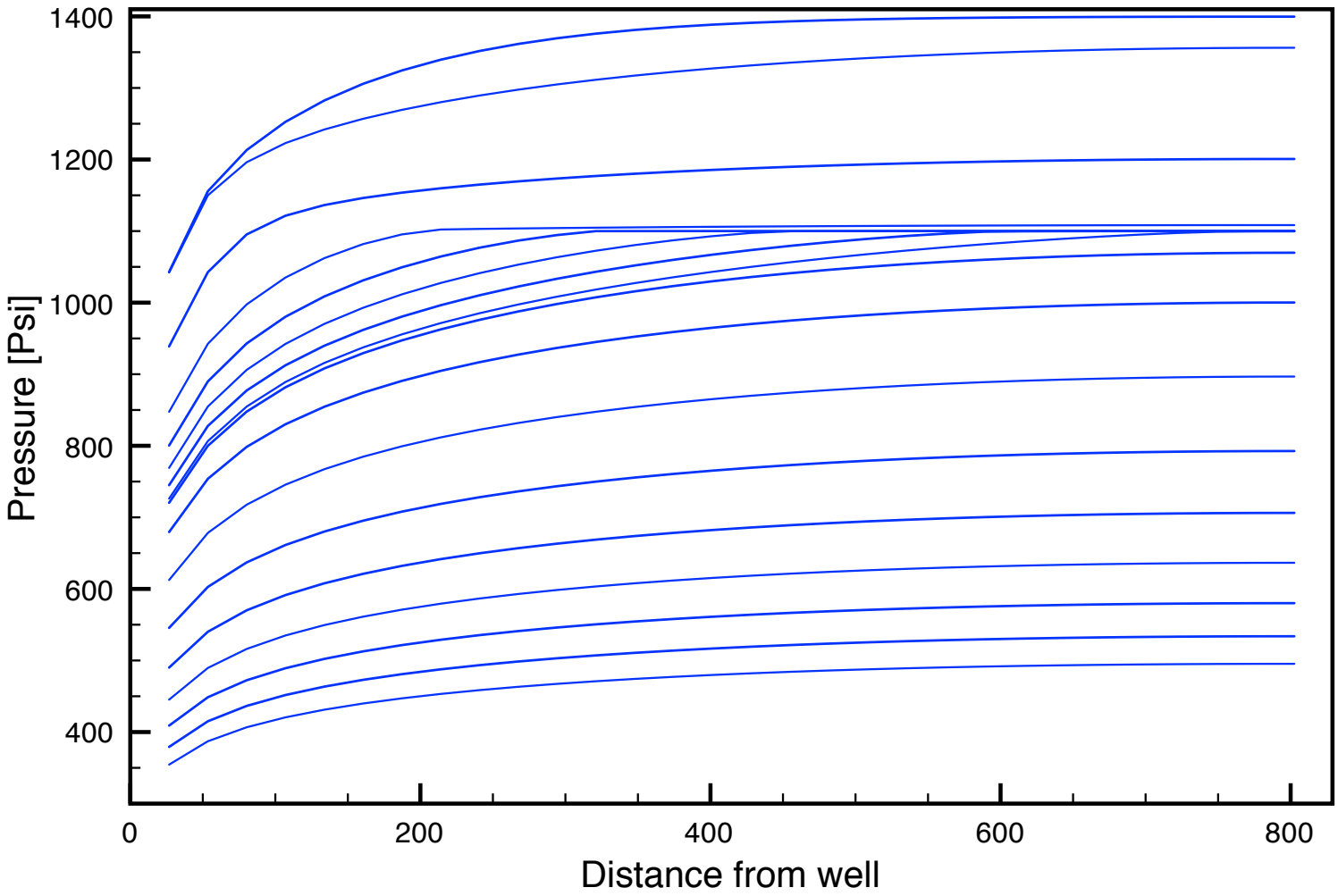
where ε is a given tolerance and θ is the maximal (given) value of $\delta t^{n+1}/\delta t^n$ or $\delta t^n/\delta t^{n+1}$.

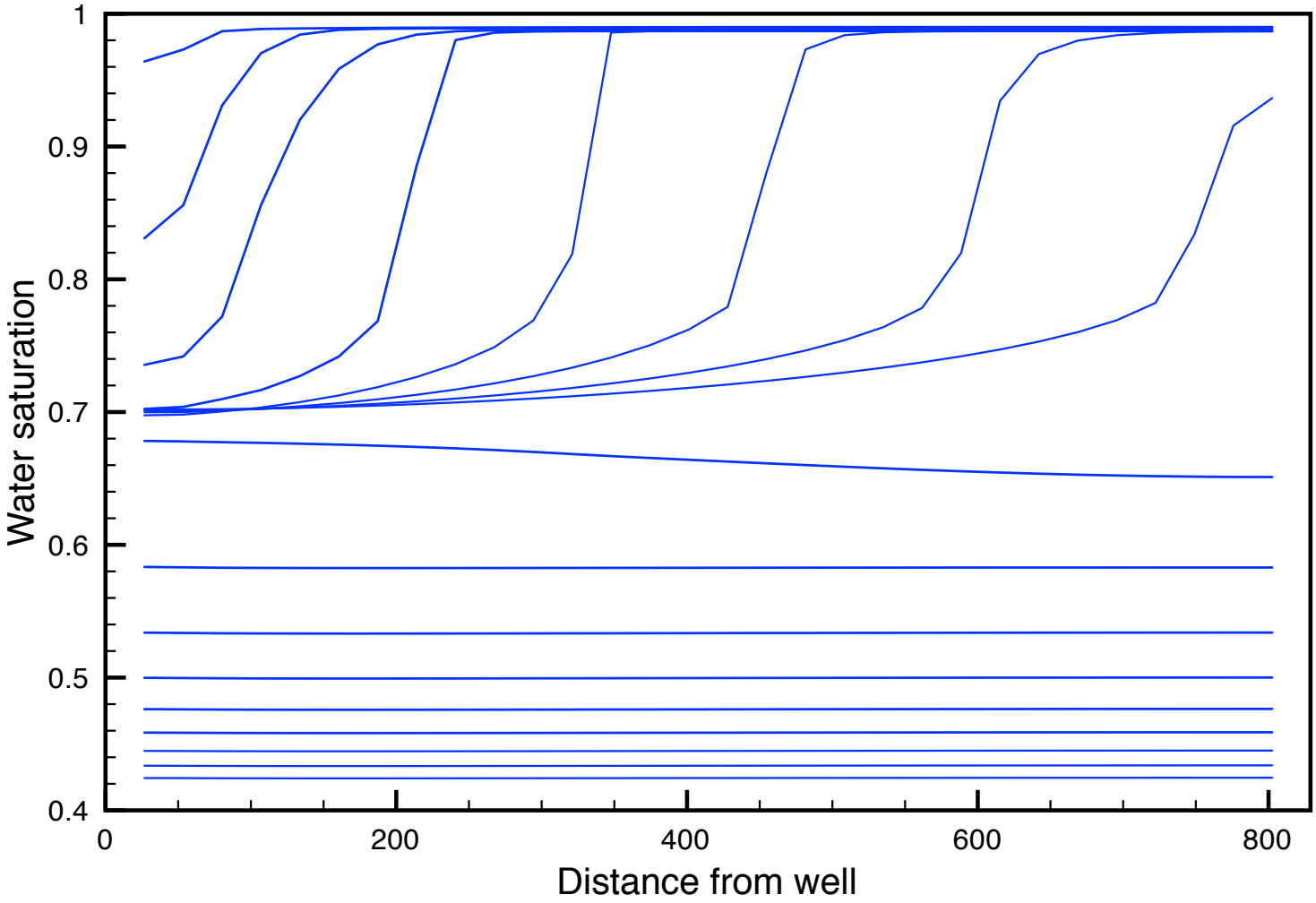
Numerical experiments: A radial case

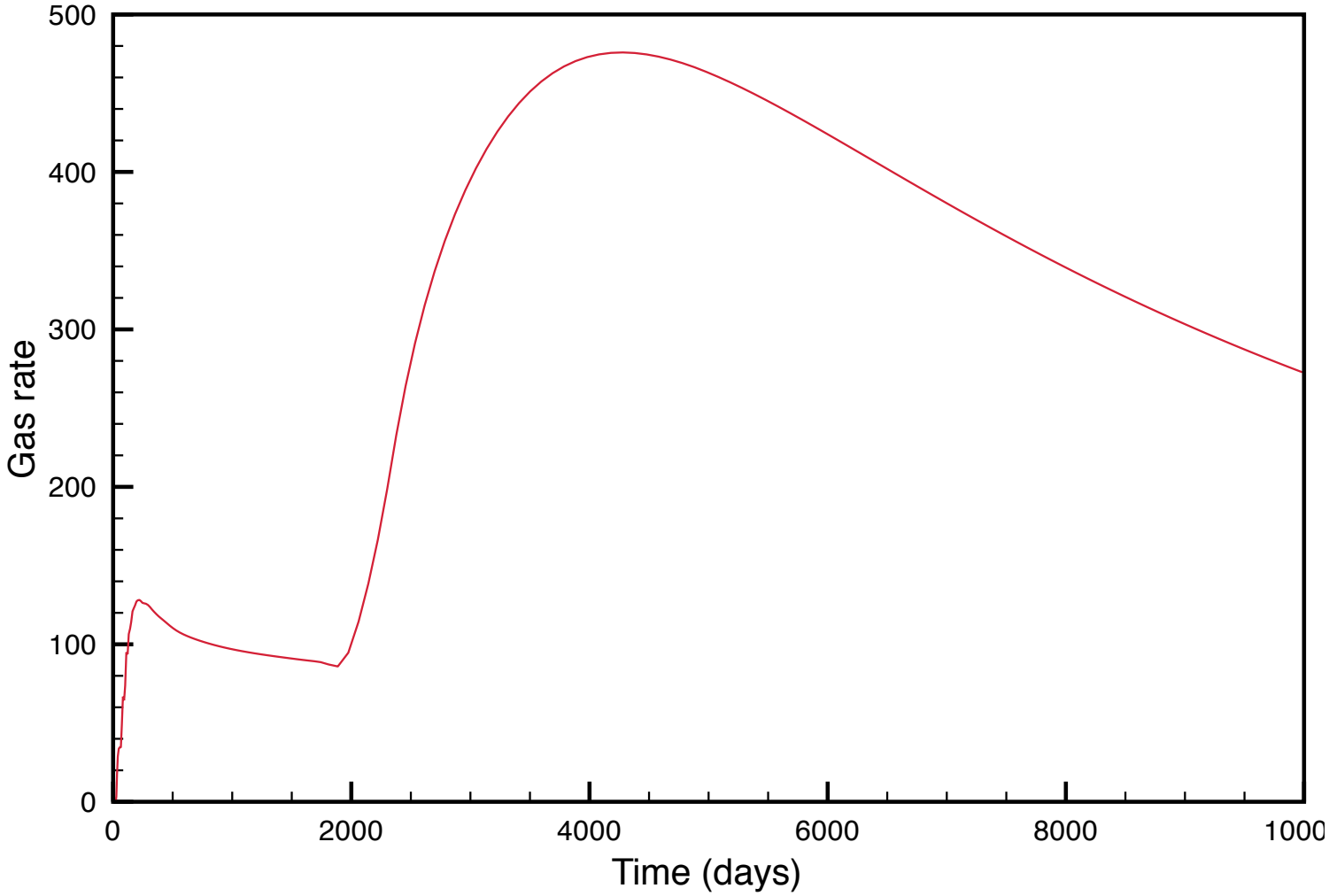
We look for a radial solution where the well is disk of radius $R_w = 0.15\,m$ located at the center of a reservoir of radius $R_e = 800\,m$, *i.e.* $R_w \ll R_e$.
We choose

$$p_c = 0, \quad S_0 = 1, \quad p_0 = 1400 \text{ psi}, \quad p_w = 100 \text{ psi}, \quad \tau = 1 \text{ day}$$

$$T_{\max} = 10000 \text{ days (more than 27 years)}$$


1400
1200
1000
800
600
400
Pressure [Psi]
0
200
400
600
800
Distance from well


Water saturation
1
0.9
0.8
0.7
0.6
0.5
0.4
0
200
400
600
800
Distance from well


500
400
300
200
100
0
Gas rate
0
2000
4000
6000
8000
1000
Time (days)

2-D Examples

Numerical simulations

- A vertical well
- A horizontal well