A stabilized finite element projection scheme for incompressible fluid flow

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Stokes Equations

Consider the time-dependent Stokes equations :

$$\frac{\partial \mathbf{u}}{\partial t} - \Delta \mathbf{u} + \nabla p = \mathbf{f} \qquad \text{in } \Omega \times (0, T)$$

$$\operatorname{div} \mathbf{u} = 0 \qquad \text{in } \Omega \times (0, T)$$

$$\mathbf{u} = 0 \qquad \text{on } \Gamma \times (0, T)$$

where $\Omega \subset \mathbb{R}^d$, $\Gamma = \partial\Omega$, $\mathbf{f} \in L^2(\Omega)^d$.

A projection scheme

$$\begin{cases} \dfrac{\widetilde{\mathbf{u}}^{n+1} - \mathbf{u}^n}{\delta t} - \Delta \widetilde{\mathbf{u}}^{n+1} = \mathbf{f}^{n+1} - \nabla p^n & \text{in } \Omega \\ \widetilde{\mathbf{u}}^{n+1} = 0 & \text{on } \Gamma \end{cases}$$

$$\begin{cases} \dfrac{\mathbf{u}^{n+1} - \widetilde{\mathbf{u}}^{n+1}}{\delta t} + \nabla (p^{n+1} - p^n) = 0 & \text{in } \Omega \\ \operatorname{div} \mathbf{u}^{n+1} = 0 & \text{in } \Omega \\ \mathbf{u}^{n+1} \cdot \mathbf{n} = 0 & \text{on } \Gamma \end{cases}$$

Remarks :

- This scheme is of first order ($O(\delta t)$)
- We have :

$$-\Delta(p^{n+1}-p^n) = -\frac{1}{\delta t}\,\mathrm{div}\,\widetilde{\mathbf{u}}^{n+1} \qquad \text{in } \Omega$$

$$\frac{\partial(p^{n+1}-p^n)}{\partial n} = 0 \qquad \text{on } \Gamma$$

Then

$$\mathbf{u}^{n+1} = \widetilde{\mathbf{u}}^{n+1} + \delta t \nabla(p^{n+1}-p^n) \qquad \text{in } \Omega$$

No inf-sup condition is required.
Solution by this formulation gives poor results :
One must first discretize in space :

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$$\frac{1}{\delta t}\mathbf{M}(\widetilde{\mathbf{u}}_h^{n+1} - \mathbf{u}_h^n) + \mathbf{A}\widetilde{\mathbf{u}}_h^{n+1} = \mathbf{b}^{n+1} - \mathbf{B}^T p_h^n$$
$$\frac{1}{\delta t}\mathbf{M}(\mathbf{u}_h^{n+1} - \widetilde{\mathbf{u}}_h^{n+1}) + \mathbf{B}^T(p_h^{n+1} - p_h^n) = 0$$
$$\mathbf{B}\mathbf{u}_h^{n+1} = 0.$$

Whence

$$\mathbf{B}\mathbf{M}^{-1}\mathbf{B}^T(p_h^{n+1} - p_h^n) = \frac{1}{\delta t}\mathbf{B}\widetilde{\mathbf{u}}_h^{n+1}$$

Remarks.

- $\mathbf{B}\mathbf{M}^{-1}\mathbf{B}^T$ is an analog of $-\mathrm{div}_h \nabla_h$ (not $-\Delta_h$!!). This matrix is sparse if a mass lumping is used.
  In general, it is less sparse than $-\Delta_h$.
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Stabilized Methods

Aim : Avoid the inf-sup condition by modifying in a consistent way the variationsl formulation

$\Rightarrow$ Possibility of using arbitrary combinations of velocity and pressure spaces.

Example : Method of Hughes, Balestra, Franca for steady state Stokes equations :

$$\begin{cases} (\nabla \mathbf{u}_h, \nabla \mathbf{v}) - (p, \operatorname{div} \mathbf{v}) = (\mathbf{f}, \mathbf{v}) & \mathbf{v} \in V_h \\ (\operatorname{div} \mathbf{u}_h, q) + \alpha \sum_T h_T^2 ((\nabla p_h, \nabla q)_T & \\ \qquad - (\Delta \mathbf{u}_h, \nabla q)_T) = \alpha \sum_T h_T^2 (\mathbf{f}, \nabla q)_T & q \in Q_h \end{cases}$$

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A Stabilisation of the projection equation

(Hughes, Masud)

The projection equation is a Darcy equation (porous media) :

Projection of $H_0^1(\Omega)^d$ on the space $\{\mathbf{v} \in L^2(\Omega)^d;\ \operatorname{div} \mathbf{v} = 0\}$:

Given $\mathbf{u} \in H_0^1(\Omega)^d$, Find $\mathbf{v} \in H(\operatorname{div};\Omega)$, $p \in L_0^2(\Omega)$ such that :

$$\begin{cases} \mathbf{v} + \nabla p = \mathbf{u} & \text{in } \Omega \\ \operatorname{div} \mathbf{v} = 0 & \text{in } \Omega \\ \mathbf{v} \cdot \mathbf{n} = 0 & \text{on } \Gamma \end{cases}$$

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Mixed Formulation :

$\mathcal{T}_h$: A mesh of Ω (triangles or tetrahedra).

$$V_h = \{\mathbf{w} \in H(\mathrm{div}, \Omega);\ \mathbf{w}_{|T} \in RT_0(T),\ \ T \in \mathcal{T}_h\}$$

$$Q_h = \{q \in L^2(\Omega);\ q_{|T} \in P_0,\ \ T \in \mathcal{T}_h,\ \int_\Omega q = 0\}$$

$$RT_0(T) = \{\mathbf{w} : T \to \mathbb{R}^2;\ \mathbf{w}(\mathbf{x}) = \mathbf{a} + b\mathbf{x},\ \mathbf{a} \in \mathbb{R}^2,\ b \in \mathbb{R}\} \quad T \in \mathcal{T}_h$$

$$\begin{cases} \text{Find } (\mathbf{v}_h, p_h) \in V_h \times Q_h \text{ such that :} & \\ (\mathbf{v}_h, \mathbf{w}) - (p_h, \operatorname{div} \mathbf{w}) = (\mathbf{u}, \mathbf{w}) & \forall\ \mathbf{w} \in V_h \\ (\operatorname{div} \mathbf{v}_h, q) = 0 & \forall\ q \in Q_h \end{cases}$$

Remark. An efficient method consists in using a mixed hybrid method (Dubois, Touzani, Zimmerman). It enables decoupling $\mathbf{v}$ and p

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Let $V^S = L^2(\Omega)^d$, $Q^S = \{q \in H^1(\Omega);\ \int_\Omega q = 0\}$.
The stabilized formulation reads :

$$\begin{cases} \text{Find } (\mathbf{v}, p) \in V^S \times Q^S \text{ such that :} \\ (\mathbf{v}, \mathbf{w}) + (\nabla p, \mathbf{w}) = (\mathbf{u}, \mathbf{w}) & \forall\, \mathbf{w} \in V^S, \\ -(\mathbf{v}, \nabla q) + (\nabla p, \nabla q) = (\mathbf{u}, \nabla q) & \forall\, q \in Q^S. \end{cases}$$

Note that this implies :

$$\mathbf{v} + \nabla p = \mathbf{u} \quad \Rightarrow \quad \operatorname{div} \mathbf{v} + \Delta p = \operatorname{div} \mathbf{u}$$
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Whence $\operatorname{div} \mathbf{v} = 0$. We also deduce $\mathbf{v} \cdot \mathbf{n} = 0$ on Γ.

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We define the forms

$$\mathscr{B}((\mathbf{v},p);(\mathbf{w},q)) = (\mathbf{v},\mathbf{w}) + (\nabla p,\mathbf{w}) - (\mathbf{v},\nabla q) + (\nabla p,\nabla q)$$
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The Lax–Milgram lemma ensures existence and uniqueness of a solution of the continuous and the discrete problems if we chose finite element spaces $V_h \subset H_0^1(\Omega)^d$ and $Q_h \subset L_0^2(\Omega)$.

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Discretization.

$$V_h^S = \{\mathbf{w} \in \mathcal{C}^0(\overline{\Omega})^d;\ \mathbf{w}_{|T} \in (P_1)^d,\ T \in \mathcal{T}_h\},$$

$$Q_h^S = \{q \in \mathcal{C}^0(\overline{\Omega});\ q_{|T} \in P_1,\ T \in \mathcal{T}_h,\ \int_\Omega q_h = 0\}$$

Convergence Analysis.

We have (Masud–Hughes) :

$$\|\mathbf{v} - \mathbf{v}_h\|_0 + \|\nabla(p - p_h)\|_0 \leq C\,(h^2\,|\mathbf{v}|_2 + h\,|p\,|_2)$$

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Implementation.

The matrix formulation reads :

$$\begin{pmatrix} \mathbf{M} & \mathbf{B} \\ -\mathbf{B}^T & \mathbf{A} \end{pmatrix} \begin{pmatrix} \mathbf{v} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{M}\,\mathbf{u} \\ \mathbf{B}^T\mathbf{u} \end{pmatrix}$$

Then

$$(\mathbf{A} + \mathbf{B}^T\mathbf{M}^{-1}\mathbf{B})\,\mathbf{p} = 2\,\mathbf{B}^T\mathbf{u}$$

$$\mathbf{v} = \mathbf{u} - \mathbf{B}\,\mathbf{p}$$

This is analogous to

$$-(\Delta_h + \mathrm{div}_h \nabla_h)p_h = -2\,\mathrm{div}\,\mathbf{u}_h$$

Remark. The matrix $\mathbf{B}^T\mathbf{M}^{-1}\mathbf{B}$ is sparse if $\mathbf{M}$ is diagonal (mass lumping).

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A stabilization of the Stokes equations

Define the spaces :

$$V_h = \{\mathbf{v} \in \mathcal{C}^0(\overline{\Omega})^d;\ \mathbf{v}_{|T} \in P_k^d,\ T \in \mathcal{T}_h,\ \mathbf{v}_{|\Gamma} = 0\}$$

$$Q_h = \{q \in \mathcal{C}^0(\overline{\Omega});\ q_{|T} \in P_\ell,\ T \in \mathcal{T}_h,\ \int_\Omega q = 0\}$$

We define a stabilized projection scheme by :

$$\begin{cases} \mathbf{u}_h^{n+1} \in V_h,\ \widetilde{\mathbf{u}}_h^{n+1} \in V_h,\ p_h^{n+1} \in Q_h \\ \dfrac{1}{\delta t}(\widetilde{\mathbf{u}}_h^{n+1} - \mathbf{u}_h^n, \mathbf{v}) + (\nabla \widetilde{\mathbf{u}}_h^{n+1}, \nabla \mathbf{v}) = (\mathbf{f}^{n+1}, \mathbf{v}) - (\nabla p_h^n, \mathbf{v}) & \mathbf{v} \in V_h \\ \dfrac{1}{\delta t}(\mathbf{u}_h^{n+1} - \widetilde{\mathbf{u}}_h^{n+1}, \mathbf{v}) + (\nabla(p_h^{n+1} - p_h^n), \mathbf{v}) = 0 & \mathbf{v} \in V_h \\ -(\mathbf{u}_h^{n+1}, \nabla q) + \delta t(\nabla(p_h^{n+1} - p_h^n), \nabla q) = (\widetilde{\mathbf{u}}_h^{n+1}, \nabla q) & q \in Q_h \end{cases}$$

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$$Q_h = \{q \in \mathcal{C}^0(\overline{\Omega});\ q_{|T} \in P_\ell,\ T \in \mathcal{T}_h,\ \int_\Omega q = 0\}$$

We define a stabilized projection scheme by :

$$\begin{cases} \mathbf{u}_h^{n+1} \in V_h,\ \widetilde{\mathbf{u}}_h^{n+1} \in V_h,\ p_h^{n+1} \in Q_h \\ \dfrac{1}{\delta t}(\widetilde{\mathbf{u}}_h^{n+1} - \mathbf{u}_h^n, \mathbf{v}) + (\nabla \widetilde{\mathbf{u}}_h^{n+1}, \nabla \mathbf{v}) = (\mathbf{f}^{n+1}, \mathbf{v}) - (\nabla p_h^n, \mathbf{v}) & \mathbf{v} \in V_h \\ \dfrac{1}{\delta t}(\mathbf{u}_h^{n+1} - \widetilde{\mathbf{u}}_h^{n+1}, \mathbf{v}) + (\nabla(p_h^{n+1} - p_h^n), \mathbf{v}) = 0 & \mathbf{v} \in V_h \\ -(\mathbf{u}_h^{n+1}, \nabla q) + \delta t(\nabla(p_h^{n+1} - p_h^n), \nabla q) = (\widetilde{\mathbf{u}}_h^{n+1}, \nabla q) & q \in Q_h \end{cases}$$

Convergence

(Dubois, Touzani)
We take $d = 2$, $k = \ell = 1$. Then, under the regularity assumptions :

$$u, u_t \in L^\infty(H^2(\Omega)^2),\ u_{tt} \in L^\infty(H^1(\Omega)^2),$$
$$p, p_t \in L^\infty(H^1(\Omega)),\ p_{tt} \in L^\infty(L^2(\Omega)),$$

we have the error bounds :

$$\|\mathbf{u} - \mathbf{u}_h\|_{\ell^\infty(H^1(\Omega)^2)} + \|p - p_h\|_{\ell^\infty(L^2(\Omega))} \leq C\,(h + \delta t),$$
$$\|\mathbf{u} - \mathbf{u}_h\|_{\ell^\infty(L^2(\Omega)^2)} \leq C\,(h^2 + \delta t).$$

This result is generalizable to the 3-D case.

Extensions

A *2nd-order* scheme for Navier-Stokes equations :

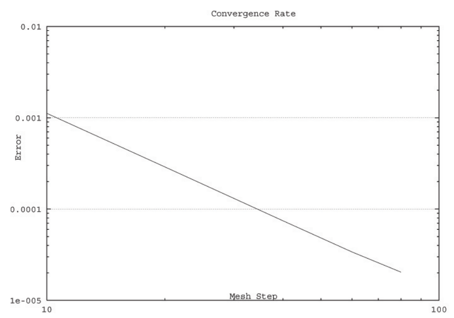
- Cranck–Nicholson for the viscosity term
- Adams–Bashforth for the explicit convective term

$(\mathbf{Mv},\mathbf{w}) := (\mathbf{v},\mathbf{w})$	Mass
$(\mathbf{Kv},\mathbf{w}) := \nu\,(\nabla\mathbf{v},\nabla\mathbf{w})$	Viscosity
$(\mathbf{C}(\mathbf{v}),\mathbf{w}) := (\mathbf{v}\cdot\nabla\mathbf{v},\mathbf{w})$	Convection
$(\mathbf{Bq},\mathbf{w}) := (\nabla q,\mathbf{w})$	Pressure gradient
$(\mathbf{Ap},\mathbf{q}) := (\nabla p,\nabla q)$	Pressure Poisson equation
$(\mathbf{b},\mathbf{w}) := (\mathbf{f},\mathbf{w})$	External forces

⇒ 2nd-order projection scheme

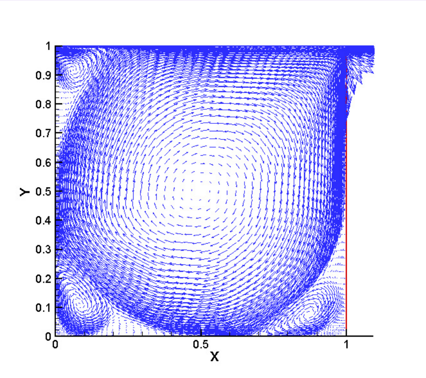
$$
\begin{aligned}
\frac{1}{\delta t}\,\mathbf{M}(\widetilde{\mathbf{u}}^{n+1} - \mathbf{u}^n) + \frac{1}{2}\mathbf{K}\,\widetilde{\mathbf{u}}^{n+1} &= \mathbf{b}^{n+\frac{1}{2}} - \mathbf{B}\,\mathbf{p}^n - \frac{1}{2}\,\mathbf{K}\,\mathbf{u}^n \\
&\quad - \frac{3}{2}\,\mathbf{C}(\mathbf{u}^n) + \frac{1}{2}\,\mathbf{C}(\mathbf{u}^{n-1}) \\
(\mathbf{A} + \mathbf{B}\mathbf{M}^{-1}\mathbf{B})\,\mathbf{q}^{n+1} &= 2\,\mathbf{B}^T\mathbf{u}^{n+1} \\
\mathbf{M}\,\mathbf{u}^{n+1} &= \mathbf{M}\,\widetilde{\mathbf{u}}^{n+1} - \mathbf{B}\,\mathbf{q}^{n+1} \\
\mathbf{p}^{n+1} &= \mathbf{p}^n + 2\delta t\,\mathbf{q}^{n+1}
\end{aligned}
$$

Numerical Test

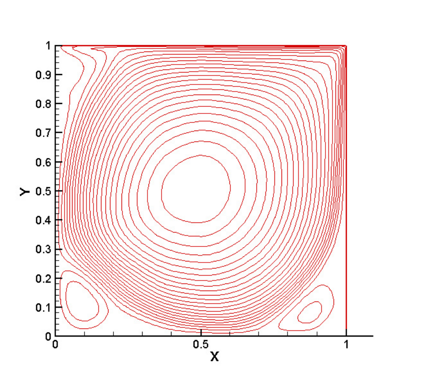


Rate ≈ 1.9

Example : Driven cavity Flow



Velocity (Re=500)



Streamlines (Re=500)